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Theoretical Computer Science 315 (2004) 319–333

Theoretical Computer Science

www.elsevier.com/locate/tcs

Bernstein–Bezoutian matrices

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Abstract

Several computational and structural properties of Bezoutian matrices expressed with respect to the Bernstein polynomial basis are shown. The exploitation of such properties allows the design of fast algorithms for the solution of Bernstein–Bezoutian linear systems without never making use of potentially ill-conditioned reductions to the monomial basis. In particular, we devise an algorithm for the computation of the greatest common divisor (GCD) of two polynomials in Bernstein form. A series of numerical tests are reported and discussed, which indicate that Bernstein–Bezoutian matrices are much less sensitive to perturbations of the coefficients of the input polynomials compared to other commonly used resultant matrices generated after having performed the explicit conversion between the Bernstein and the power basis.

Keywords: Bezoutian matrices; Bernstein polynomial basis; Displacement structure; Fast algorithms

1. Introduction

Approximation methods based on Bézier curves have become more and more popular in computer aided geometric design (CAGD) [10–12,14,27]. Since Bézier curves are parametrized by means of Bernstein polynomials, it follows that computational problems involving Bézier curves generally reduce to manipulating polynomials expressed with respect to the Bernstein polynomial basis. In particular, Bézier curve intersection problems are shown to be equivalent to checking the relative primality of two polynomials in the Bernstein basis. Explicit conversion between the Bernstein and the power polynomial basis is exponentially ill-conditioned as the polynomial degree n increases [13]. Therefore, for numerical computations involving polynomials in Bernstein form

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doi:10.1016/j.tcs.2004.01.016

it is essential to consider algorithms which express all intermediate results using this form only.

The purpose of this paper is to provide theoretical bases for the design of fast and accurate algorithms for computing the greatest common divisor (GCD) of two real polynomials $p(z)$ and $q(z)$ of degree at most n expressed in the Bernstein polynomial basis $\{\beta_0^{(n)}(z),\dots,\beta_n^{(n)}(z)\}$, where $\beta_i^{(n)}(z)=\binom{n}{i}(1-z)^{n-i}z^i$, $0\leqslant i\leqslant n$. In theory, fast $O(n^2)$ algorithms can be obtained by first determining the power form of $p(z)$ and $q(z)$ and then by applying some method based on the subresultant theory [7,5] or its matrix counterparts [3] to evaluate their GCD. However, due to the ill-conditioning of the explicit conversion between the Bernstein and the power basis, it has been shown [24,25] that such an approach may suffer from severe numerical difficulties and, in particular, the worst case precision of $O(n)$ bits is nearly always required in calculations to retain some significant correct bits in the output.

In this paper, we circumvent these difficulties by considering a modified resultant matrix for polynomials in Bernstein form which is represented by its short displacement generator. This displacement representation is novel and quite efficient, being explicit, algebraic and available at practically no cost. That is, we have another important example where algebraic techniques come to rescue to overcome numerical difficulty.

A solution of the GCD problem for polynomials in the Bernstein form, which does not employ any basis conversion, is first provided in [24]. The approach relies upon the construction of a suitable Frobenius matrix $F\in\mathbb{R}^{n\times n}$ of $p(z)$ directly determined from the coefficients of its representation in the Bernstein basis. Given such a matrix F, one can consider the matrix $q(F)$ obtained by evaluating the polynomial $q(z)$ at the matrix F. This matrix inherits several properties of the resultant matrix of $p(z)$ and $q(z)$ and, in particular, its LU factorization yields the coefficients of the GCD of $p(z)$ and $q(z)$. The results of numerical experiments presented in [26] show that $q(F)$ is generally better conditioned than $q(C)$, where C is the classical Frobenius matrix associated with $p(z)$. This improvement of the accuracy is, however, paid for by an increase of the computational cost of the resulting method. The calculation of the entries of $q(F)$, as outlined in [24], given the entries of F and the coefficients of $q(z)$ in the Bernstein basis has a cost of $O(n^3)$ arithmetic operations (ops). In addition to that, the factorization phase, where Gaussian elimination is applied to $q(F)$ in order to reduce $q(F)$ in its row echelon form, also requires $O(n^3)$ ops.

In this paper, we introduce the Bezout form $B=(b_{i,j})\in\mathbb{R}^{n\times n}$ of the resultant of $p(z)$ and $q(z)$ defined by

$$\frac{p(z)q(w)-p(w)q(z)}{z-w}=\sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w).$$

Bezoutian matrices with respect to different polynomial bases have been previously considered by many authors (see [1,16,18,19,23]). Quite apart from their theoretical interest, they have been proved to be a powerful tool for devising efficient numerical methods for computations with polynomials and structured matrices [4].

In Section 2, we show that the matrix B can be constructed using $O(n^2)$ ops given the coefficients expressing $p(z)$ and $q(z)$ in the Bernstein basis $\{\beta_0^{(n)}(z),\dots,\beta_n^{(n)}(z)\}$.

In addition to that, we relate the properties of a block triangular factorization of B with the ones of a certain polynomial remainder algorithm applied to the reversed polynomials of $p(z)$ and $q(z)$. This result enables the computation of the GCD of $p(z)$ and $q(z)$ to be reduced either to computing a block LU factorization of B or to solving an homogeneous linear system with coefficient matrix being the kth leading principal submatrix of B for a suitable k.

Since we are interested in using floating point arithmetic, it is worth realizing that, independently of the numerical method we consider to solve these problems, the precision of computations must be dynamically tuned according to the condition number of the leading principal submatrices of B. For input polynomials in Bernstein form we have performed extensive numerical experiments, which are partly reported and discussed in Section 4, by comparing the conditioning profile of B with that one of the classical Bezout matrix $\tilde{B}$ generated after having explicitly evaluated the coefficients of the polynomials in the monomial basis. Almost in any case the conditioning profile of B was significantly better than the one of $\tilde{B}$ whereas in the remaining few cases they were comparable. Similar conclusions are reached in [26] for a different resultant matrix for polynomials in Bernstein form. Hence, the Bezout–resultant matrix B is numerically superior to its power basis equivalent.

Among the numerical algorithms that solve underdetermined linear systems, it is known that SVD provides the most reliable one. Methods based on SVD computations on subresultant matrices for numerically computing GCDs of polynomials in power form have been proposed in [8,9] (see also [22] for a discussion on these methods compared with some known approaches as well as for extensions to other resultant matrices). These methods can be generalized to polynomials in Bernstein form by simply considering a different matrix formulation relying on the use of the matrix B.

An alternative approach exploiting the reduction of the GCD computation to the block LU factorization of B is motivated by the structural properties of the resultant matrix B. In Section 3, we describe the displacement structure of B by proving that $\mathscr{F}(B)$ is a small rank matrix, say $\mathscr{F}(B)=\sum_{i=1}^{r} \boldsymbol{u}_i\boldsymbol{v}_i^{\mathrm{T}}$ with $r \ll n$, where

$$\mathscr{F}:\ \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad \mathscr{F}(B)=L^{\mathrm{T}}B-BL$$

and $L \in \mathbb{R}^{n\times n}$ is a lower bidiagonal matrix with unit diagonal entries. The vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are called generators of the displacement representation of B. The displacement structure of B can be incorporated into the calculations of its block triangular factorization. In particular, we find that a suitable variant of the block Gaussian elimination scheme only using recursions on the generators can be applied to B thus leading to a fast $\mathrm{O}(n^2)$ algorithm for the computation of the GCD of $p(z)$ and $q(z)$. This algorithm can be made robust in floating point arithmetic by replacing zero-check conditions with criteria based both on backward error analysis for LU factorization and conditioning estimates for the leading principal submatrices of B. Fast numerical schemes based on similar techniques were developed in [6,15] for the solution of Toeplitz and Hankel linear systems. The generalization of the error analysis presented there to Bernstein–Bezoutian linear systems is beyond our present scope and it is a part of an ongoing investigation on the numerical properties of resultants for Bernstein polynomials.

Section 5 contains a brief discussion on future work that follows on from the results described in this paper.

2. Bezoutians of polynomials in Bernstein form

In this section, we introduce the Bezout form of the resultant of two polynomials expressed in the Bernstein basis by showing that its properties can be used to compute the greatest common divisor (GCD) of polynomials in Bernstein form.

Let $p(z)$ and $q(z)$ be two real polynomials of degree less than or equal to n. The polynomials $\beta_i^{(k)}(z) = \binom{k}{i}(1-z)^{k-i}z^i$, $0 \leqslant i \leqslant k$, form the Bernstein basis of the vector space of real polynomials of degree at most k. Assume that

$$p(z) = \sum_{i=0}^{n} p_i \beta_i^{(n)}(z), \quad q(z) = \sum_{i=0}^{n} q_i \beta_i^{(n)}(z) \tag{1}$$

defines the Bernstein form of $p(z)$ and $q(z)$, respectively. From

$$z^j = \sum_{k=j}^{n} \binom{k}{j} \binom{n}{j}^{-1} \beta_k^{(n)}(z), \quad j = 0, \ldots, n,$$

one immediately obtains that the matrix $T_n = (t_{i,j}^{(n)}) \in \mathbb{R}^{(n+1)\times(n+1)}$ defining the transformation between the Bernstein and the power basis is given by

$$T_n \begin{bmatrix} \beta_0^{(n)}(z) \\ \vdots \\ \beta_n^{(n)}(z) \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ z^n \end{bmatrix}, \quad t_{i,j}^{(n)} = \begin{cases} 0 & \text{if } i > j, \\ \binom{j-1}{i-1}\binom{n}{i-1}^{-1} & \text{if } i \leqslant j. \end{cases} \tag{2}$$

The Bezoutian matrix $B = (b_{i,j}) \in \mathbb{R}^{n\times n}$ of $p(z)$ and $q(z)$ in the Bernstein basis $\{\beta_0^{(n-1)}(z), \ldots, \beta_n^{(n-1)}(z)\}$ is defined by

$$\frac{p(z)q(w) - p(w)q(z)}{z - w} = \sum_{i,j=1}^{n} b_{i,j} \beta_{i-1}^{(n-1)}(z) \beta_{j-1}^{(n-1)}(w) \tag{3}$$

which can equivalently be rewritten as

$$\frac{p(z)q(w) - p(w)q(z)}{z - w} = [\beta_0^{(n-1)}(z), \ldots, \beta_{n-1}^{(n-1)}(z)] B \begin{bmatrix} \beta_0^{(n-1)}(w) \\ \vdots \\ \beta_{n-1}^{(n-1)}(w) \end{bmatrix}. \tag{4}$$

Our first result is concerned with the construction of the matrix B given the coefficients of the Bernstein form (1) of $p(z)$ and $q(z)$.

Theorem 1. *Given the coefficients p_i, q_i, $0 \leqslant i \leqslant n$, as in (1), the Bernstein–Bezoutian matrix $B = (b_{i,j}) \in \mathbb{R}^{n\times n}$ satisfying (3) can be constructed at the cost of* $O(n^2)$

arithmetic operations according to the following rules:

$$b_{i,1} = \frac{n}{i}\,(p_i q_0 - p_0 q_i), \quad 1 \leqslant i \leqslant n,$$

$$b_{i,j+1} = \frac{n^2}{i(n-j)}\,(p_i q_j - p_j q_i) + \frac{j(n-i)}{i(n-j)} b_{i+1,j}, \quad 1 \leqslant i,j \leqslant n-1,$$

$$b_{n,j+1} = \frac{n}{n-j}\,(p_n q_j - p_j q_n), \quad 1 \leqslant j \leqslant n-1.$$

Proof. From (1) and (3) we deduce that

$$\sum_{i,j=0}^{n} (p_i q_j - p_j q_i)\beta_i^{(n)}(z)\beta_j^{(n)}(w) = (z-w)\sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w).$$

Since

$$\begin{aligned} z &\sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w) \\ &= (w + (1-w))z \sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w) \\ &= zw \sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w) + \sum_{i,j=1}^{n} b_{i,j} z\beta_{i-1}^{(n-1)}(z)(1-w)\beta_{j-1}^{(n-1)}(w) \end{aligned}$$

and, similarly,

$$\begin{aligned} w &\sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w) \\ &= (z + (1-z))w \sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w) \\ &= zw \sum_{i,j=1}^{n} b_{i,j}\beta_{i-1}^{(n-1)}(z)\beta_{j-1}^{(n-1)}(w) + \sum_{i,j=1}^{n} b_{i,j}(1-z)\beta_{i-1}^{(n-1)}(z)w\beta_{j-1}^{(n-1)}(w), \end{aligned}$$

one finds that

$$\begin{aligned} &\sum_{i,j=0}^{n} (p_i q_j - p_j q_i)\beta_i^{(n)}(z)\beta_j^{(n)}(w) \\ &\quad = \sum_{i,j=1}^{n} b_{i,j} z\beta_{i-1}^{(n-1)}(z)(1-w)\beta_{j-1}^{(n-1)}(w) + - \sum_{i,j=1}^{n} b_{i,j}(1-z)\beta_{i-1}^{(n-1)}(z)w\beta_{j-1}^{(n-1)}(w). \end{aligned}$$

This can be rewritten as

$$\begin{aligned} &\sum_{i,j=0}^{n} (p_i q_j - p_j q_i)\beta_i^{(n)}(z)\beta_j^{(n)}(w) \\ &\quad = \sum_{i,j=1}^{n} b_{i,j}\,\frac{i}{n}\,\beta_i^{(n)}(z)\frac{n-j+1}{n}\,\beta_{j-1}^{(n)}(w) + - \sum_{i,j=1}^{n} b_{i,j}\,\frac{n-i+1}{n}\,\beta_{i-1}^{(n)}(z)\frac{j}{n}\,\beta_j^{(n)}(w). \end{aligned}$$

Hence, by equalizing the coefficients of $\beta_j^{(n)}(w)$ on both sides of the previous relation, it follows that

$$\sum_{i=0}^{n}(p_iq_0 - p_0q_i)\beta_i^{(n)}(z) = \sum_{i=1}^{n} b_{i,1}\,\frac{i}{n}\,\beta_i^{(n)}(z)$$

and

$$\sum_{i=0}^{n}(p_iq_j - p_jq_i)\beta_i^{(n)}(z) = \frac{n-j}{n}\sum_{i=1}^{n} b_{i,j+1}\frac{i}{n}\beta_i^{(n)}(z) - \frac{j}{n}\sum_{i=1}^{n} b_{i,j}\frac{n-i+1}{n}\beta_{i-1}^{(n)}(z)$$

for $j=1,\dots,n-1$. A comparison of the coefficients of $\beta_i^{(n)}(z)$ now concludes the proof of the theorem. $\square$

Next result relates the block LU factorization of B with the computation of the GCD of $p(z)$ and $q(z)$ expressed in the Bernstein form (1). The crucial observation is that B is congruent to the classical Bezoutian matrix associated with $p(z)$ and $q(z)$. That is, from (2) and (4) one obtains

$$\frac{p(z)q(w)-p(w)q(z)}{z-w} = [1,\dots,z^{n-1}]T_{n-1}^{-\mathrm{T}}BT_{n-1}^{-1}\begin{bmatrix}1\\ \vdots\\ w^{n-1}\end{bmatrix} \tag{5}$$

and thus $\hat{B}=T_{n-1}^{-\mathrm{T}}BT_{n-1}^{-1}$ is the classical Bezout matrix of order n associated with $p(z)$ and $q(z)$ of degree at most n.

Let $J_n\in\mathbb{R}^{n\times n}$ be the permutation (reversion) matrix having unit antidiagonal entries. Moreover, introduce the reverse polynomials $\tilde{p}(z)=z^n p(z^{-1})$ and $\tilde{q}(z)=z^n q(z^{-1})$. By multiplying both sides of

$$\frac{p(z^{-1})q(w^{-1})-p(w^{-1})q(z^{-1})}{z^{-1}-w^{-1}} = [1,\dots,z^{-n+1}]\hat{B}\begin{bmatrix}1\\ \vdots\\ w^{-n+1}\end{bmatrix},$$

by $z^{n-1}w^{n-1}$ it is readily verified that

$$\frac{\tilde{p}(z)\tilde{q}(w)-\tilde{p}(w)\tilde{q}(z)}{w-z} = [1,\dots,z^{n-1}]J_n\hat{B}J_n\begin{bmatrix}1\\ \vdots\\ w^{n-1}\end{bmatrix}$$

which says that, up to the sign, $\tilde{B}=J_n\hat{B}J_n$ is the classical Bezout matrix generated by $\tilde{p}(z)$ and $\tilde{q}(z)$.

The characterization of the Euclidean algorithm applied to the polynomials $\tilde{p}(z)$ and $\tilde{q}(z)$ in terms of properties of the block LU factorization of $\hat{B}=J_n\tilde{B}J_n$ provided in [2,3,17] enables us to show that B is indeed a resultant matrix for the polynomials $p(z)$ and $q(z)$. Given two polynomials $p(z)$ and $q(z)$ of degree at most n we say that ∞ is a common root of $p(z)$ and $q(z)$ if $\deg(p(z))<n$ and $\deg(q(z))<n$. In the following

theorem we extend the results of [2,3,17] to the representation of polynomials in the Bernstein basis.

Theorem 2. *Assume that both* 0 *and* ∞ *is not a common root of the two real polynomials* $p(z)$ *and* $q(z)$ *defined by* (1). *Moreover, let* $w(z)$ *be the GCD of* $p(z)$ *and* $q(z)$. *Then*:

(1) *The degree* k *of* $w(z)$ *is equal to* $k=n-\operatorname{rank}(B)$, *where* B *is the Bernstein–Bezoutian matrix generated from* $p(z)$ *and* $q(z)$ *as in* (3).

(2) *We have* $\det(B_{n-k})\neq 0$ *and* $\det(B_{n-j})=0$ *for* $j=k+1,\dots,n$, *where* B_j *denotes the* $j\times j$ *leading principal submatrix of* B. *Set* $1\leqslant m_1<m_2<\cdots<m_L=n-k$ *be the integers such that* $\det(B_{m_i})\neq 0$, $1\leqslant i\leqslant L$, $\det(B_j)=0$, *otherwise*.

(3) *Let* B *be partitioned into a* 2×2 *block matrix as follows*:

$$B=\begin{bmatrix} B_{m_{L-1}} & P \\ Q & R \end{bmatrix}.$$

Moreover, consider the Schur complement $S=R-QB_{m_{L-1}}^{-1}P$ *of* $B_{m_{L-1}}$ *in* B *and let* $[b_{m_{L-1}+1},\dots,b_n]$ *be the first row of* S. *There exists a nonzero scalar* α *such that*

$$b_{m_{L-1}+1}\beta_{m_{L-1}}^{(n-1)}(z)+\cdots+b_n\beta_{n-1}^{(n-1)}(z)=\alpha z^{m_{L-1}}z^k w(z^{-k}). \tag{6}$$

Proof. Since

$$J_n\tilde{B}J_n=T_{n-1}^{-\mathrm{T}}BT_{n-1}^{-1}$$

and, moreover, T_{n-1}^{T} is a nonsingular lower triangular matrix, then (1) and (2) can be easily obtained from the analogous properties of classical Bezoutians stated in [2, Corollary 3.1]. Concerning part (3), we recall that the Schur complement $\hat{S}$ of the leading principal submatrix of order m_{L-1} of $J_n\tilde{B}J_n$ is such that its first row gives the suitably normalized coefficients of the greatest common divisor $z^k w(z^{-1})$ of $\tilde{p}(z)$ and $\tilde{q}(z)$ [2,3,17]. This way, relation (6) now follows from $\hat{T}^{\mathrm{T}}\hat{S}\hat{T}=S$, where $\hat{T}$ denotes the trailing principal submatrix of T_{n-1} of order $n-m_{L-1}$. $\square$

Example 3. Consider the polynomials

$$p(z)=4-5z^2+z^4=4\beta_0^{(4)}(z)+4\beta_1^{(4)}(z)+\tfrac{19}{6}\beta_2^{(4)}(z)+\tfrac{3}{2}\beta_3^{(4)}(z)$$

and $q(z)=\frac{1}{2}-\frac{1}{4}z-2z^2+z^3$,

$$q(z)=\tfrac{1}{2}\beta_0^{(4)}(z)+\tfrac{7}{16}\beta_1^{(4)}(z)+\tfrac{1}{24}\beta_2^{(4)}(z)-\tfrac{7}{16}\beta_3^{(4)}(z)-\tfrac{3}{4}\beta_4^{(4)}(z),$$

whose (monic) greatest common divisor is $w(z)=z-2$. We find that

$$B=\begin{bmatrix} 1 & 17/6 & 10/3 & 3 \\ 17/6 & 157/36 & 83/18 & 4 \\ 10/3 & 83/18 & 187/36 & 19/4 \\ 3 & 4 & 19/4 & 9/2 \end{bmatrix}.$$

Hence, the Schur complement S of B_2 in B is

$$S = \begin{bmatrix} 5/11 & 15/22 \\ 15/22 & 45/44 \end{bmatrix}$$

and thus it is readily verified that

$$\frac{5}{11}\,\beta_2^{(3)}(z) + \frac{15}{22}\,\beta_3^{(3)}(z) = -\frac{15}{22}\,z^2(z-2).$$

The Schur complement of B_3 is the zero matrix of order 1.

In view of the triangular structure of T_{n-1}, it can also be shown that the sequence $\{m_1,\dots,m_L\}$ in Theorem 2, which corresponds the sequence of jumps in the block triangular factorization process applied to B, can be determined by means of a direct inspection of the entries of the computed Schur complements. In particular, the occurrence of a jump is revealed by zero entries in the north-western corner of S.

Example 4. Let $p(z) = 1 + 4z^4 + z^5$ and $q(z) = z + z^5$ so that

$$\tilde{p}(z) = z^5\,p(z^{-1}) = z\tilde{q}(z) + 3z + 1.$$

The Schur complement S of B_1 is

$$S = \begin{bmatrix} 0 & 0 & 3/16 & 1 \\ 0 & 1/12 & 2/3 & 8/3 \\ 3/16 & 2/3 & 17/8 & 6 \\ 1 & 8/3 & 6 & 10 \end{bmatrix}$$

and thus $m_2 = 4$.

Finally, we observe that the assumptions of Theorem 2 could be relaxed and similar properties are still valid in the degenerate cases where $p(z)$ and $q(z)$ have a common root at 0 or ∞. This situation can easily be detected by evaluating the polynomials and the reverse polynomials at the origin and, then, Theorem 2 applies to the possibly deflated polynomials.

3. The displacement structure of Bernstein–Bezoutian matrices

So far we have shown that the solution of the GCD problem for polynomials expressed with respect to the Bernstein polynomial basis can be reduced to the computation of a block triangular factorization of a certain matrix B generated according to Theorem 1 from the coefficients of these polynomials. In order to design a fast algorithm for this latter task, in this section we investigate the displacement structure of B. Next result provides a characterization of the Bernstein-companion matrix $T_{n-1}^{-1}ZT_{n-1}$, where $Z = (z_{i,j})$ is the down-shift matrix of order n defined by $z_{i,j} = \delta_{i-1,j}$ and $\delta_{i,j}$ is the Kronecker symbol.

Theorem 5. *We have*

$$T_{n-1}^{-1} Z T_{n-1} = V = \begin{bmatrix} 1+\eta_1 & \eta_2 & \dots & \eta_n \\ \gamma_1 & 1 & & \circ \\ & \ddots & \ddots & \\ \circ & & \gamma_{n-1} & 1 \end{bmatrix},$$

where $\gamma_i = (n-i)/i$, $1 \leqslant i \leqslant n-1$, $\eta_i = -n/i$, $1 \leqslant i \leqslant n$.

Proof. From (2) and from

$$z^{-1} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & & & \circ \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ \circ & & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{bmatrix} + z^{-1} \boldsymbol{e}_1$$

one finds that, for any $z \in \mathbb{R}$,

$$z^{-1} \begin{bmatrix} \beta_0^{(n-1)}(z) \\ \beta_1^{(n-1)}(z) \\ \vdots \\ \beta_{n-1}^{(n-1)}(z) \end{bmatrix} = T_{n-1}^{-1} Z T_{n-1} \begin{bmatrix} \beta_0^{(n-1)}(z) \\ \beta_1^{(n-1)}(z) \\ \vdots \\ \beta_{n-1}^{(n-1)}(z) \end{bmatrix} + z^{-1} \boldsymbol{e}_1. \tag{7}$$

From the definition of the Bernstein polynomials $\beta_i^{(n-1)}(z)$ it is easily verified that, for $1 \leqslant i \leqslant n-1$,

$$z^{-1}\beta_i^{(n-1)}(z) - \beta_i^{(n-1)}(z) = \binom{n-1}{i} (1-z)^{n-i} z^{i-1} = \frac{n-i}{i} \beta_{i-1}^{(n-1)}(z). \tag{8}$$

Moreover, since Bernstein polynomials define a partition of unity, that is, $\sum_{i=0}^{n-1} \beta_i^{(n-1)}(z) = 1$, it follows that $z^{-1} \sum_{i=0}^{n-1} \beta_i^{(n-1)}(z) = z^{-1}$, and, therefore,

$$\begin{aligned} z^{-1}\beta_0^{(n-1)}(z) &= z^{-1} - z^{-1} \sum_{i=1}^{n-1} \beta_i^{(n-1)}(z) \\ &= z^{-1} - (n-1)\beta_0^{(n-1)}(z) - \sum_{i=1}^{n-2} \left(1 + \frac{n-i-1}{i+1}\right) \beta_i^{(n-1)}(z) - \beta_{n-1}^{(n-1)}(z). \end{aligned} \tag{9}$$

Hence, by combining relations (8) and (9), we deduce that (7) still holds if we replace $T_{n-1}^{-1} Z T_{n-1}$ by the matrix V. Since the value of z can be arbitrarily chosen and $\beta_0^{(n-1)}(z), \dots, \beta_{n-1}^{(n-1)}(z)$ are linearly independent, then we may conclude that $V = T_{n-1}^{-1} Z T_{n-1}$. $\square$

From the previous theorem it is immediately seen that

$$T_{n-1}^{-1}ZT_{n-1} = L + \boldsymbol{e}_1[\eta_1, \eta_2, \dots, \eta_n],$$

where $L = I + Z\,\mathrm{diag}[\gamma_1, \dots, \gamma_n] \in \mathbb{R}^{n\times n}$ denotes the lower bidiagonal matrix with unit diagonal entries and subdiagonal entries equal to $\gamma_1, \dots, \gamma_{n-1}$. This implies the matrix equation

$$ZT_{n-1} - T_{n-1}L = \boldsymbol{e}_1[\eta_1, \eta_2, \dots, \eta_n] = \boldsymbol{e}_1\boldsymbol{\eta}^{\mathrm{T}}, \tag{10}$$

which can be used to derive a displacement equation for the Bernstein–Bezoutian matrix B generated from the coefficients of two polynomials $p(z)$ and $q(z)$ as in (3).

Recall that classical Bezoutians are the inverse of Hankel matrices and, therefore, they are Hankel-like matrices [20,21]. In particular, the Bezout matrix

$$\hat{B} = T_{n-1}^{-\mathrm{T}} B T_{n-1}^{-1} \tag{11}$$

of $p(z)$ and $q(z)$ with respect to the standard power basis satisfies

$$Z^{\mathrm{T}}\hat{B} - \hat{B}Z = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} - \boldsymbol{v}\boldsymbol{u}^{\mathrm{T}} \tag{12}$$

for two suitable $n-$dimensional vectors $\boldsymbol{u} = \hat{B}\boldsymbol{e}_1$ and $\boldsymbol{v}$. Thus, from (10) and (11) one obtains that

$$\begin{aligned} L^{\mathrm{T}}B - BL &= L^{\mathrm{T}}T_{n-1}^{\mathrm{T}}T_{n-1}^{-\mathrm{T}}B - BT_{n-1}^{-1}T_{n-1}L \\ &= (T_{n-1}^{\mathrm{T}}Z^{\mathrm{T}} - \boldsymbol{\eta}\boldsymbol{e}_1^{\mathrm{T}})\hat{B}T_{n-1} - T_{n-1}^{\mathrm{T}}\hat{B}(ZT_{n-1} - \boldsymbol{e}_1\boldsymbol{\eta}^{\mathrm{T}}), \end{aligned}$$

from which, in the light of (12), it follows that

$$\begin{aligned} L^{\mathrm{T}}B - BL &= T_{n-1}^{\mathrm{T}}(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} - \boldsymbol{v}\boldsymbol{u}^{\mathrm{T}})T_{n-1} - \boldsymbol{\eta}\boldsymbol{u}^{\mathrm{T}}T_{n-1} + T_{n-1}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{\eta}^{\mathrm{T}} \\ &= T_{n-1}^{\mathrm{T}}\boldsymbol{u}(\boldsymbol{\eta}^{\mathrm{T}} + \boldsymbol{v}^{\mathrm{T}}T_{n-1}) - (\boldsymbol{\eta} + T_{n-1}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{u}^{\mathrm{T}}T_{n-1}. \end{aligned}$$

This means that the matrix B has displacement rank at most 2 with respect to the displacement operator

$$\mathscr{F}\colon\ \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad \mathscr{F}(B) = L^{\mathrm{T}}B - BL$$

or, equivalently, $\mathrm{rank}(\mathscr{F}(B)) \leqslant 2$. By looking back at Theorem 1, we find that, for $1 \leqslant i,j \leqslant n-1$,

$$\begin{aligned} (L^{\mathrm{T}}B - BL)_{i,j} &= \gamma_i b_{i+1,j} - \gamma_j b_{i,j+1} \\ &= \frac{n-j}{j}\left(\frac{j(n-i)}{i(n-j)}\, b_{i+1,j} - b_{i,j+1}\right) = -\frac{n^2}{ij}\,(p_i q_j - p_j q_i). \end{aligned}$$

Analogously for $j = 1, \dots, n$, we obtain that

$$(L^{\mathrm{T}}B - BL)_{n,j} = -\gamma_j b_{n,j+1} = -\frac{n}{j}\,(p_n q_j - p_j q_n).$$

Observe that $L^{\mathrm{T}}B - BL = \tilde{L}^{\mathrm{T}}B - B\tilde{L}$ for $\tilde{L} = L - I = DZD^{-1}$, where $D = \mathrm{diag}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n-1}\right)$. That is, the scaled Bezoutian matrix $\tilde{B} = DBD$ is such that

$Z^{\mathrm{T}}\tilde{B} - \tilde{B}Z$ has rank at most 2. Observe also that, if p_i and q_j are integers, then the scaled Bezoutian $\tilde{B}$ as well as the matrix $Z^{\mathrm{T}}\tilde{B} - \tilde{B}Z$ have integer entries. The latter matrix can be written as $(Z^{\mathrm{T}}\tilde{B} - \tilde{B}Z)_{i,j} = (n^2/ij)d_i d_j(p_i q_j - p_j q_i)$, where $d_i = \binom{n}{i-1}$.

In this way, we arrive at the following result, which characterizes the generators of the displacement representation of B in terms of the coefficients of the Bernstein form of $p(z)$ and of $q(z)$.

Theorem 6. *The Bernstein–Bezoutian matrix B generated from $p(z)$ and $q(z)$ by means of (3) satisfies the displacement equation*

$$L^{\mathrm{T}}B - BL = \hat{\boldsymbol{q}}\hat{\boldsymbol{p}}^{\mathrm{T}} - \hat{\boldsymbol{p}}\hat{\boldsymbol{q}}^{\mathrm{T}},$$

where $L = I + Z\,\mathrm{diag}[\gamma_1, \ldots, \gamma_n]$, $\hat{\boldsymbol{p}} = [np_1, (n/2)p_2, \ldots, p_n]^{\mathrm{T}}$, $\hat{\boldsymbol{q}} = [nq_1, (n/2)q_2, \ldots, q_n]^{\mathrm{T}}$ and p_i, q_i, $0 \leqslant i \leqslant n$, are the coefficients of the Bernstein form (1) of $p(z)$ and $q(z)$, respectively.

If J_n denotes the reversion matrix introduced in the previous section, then Theorem 6 provides a displacement equation for J_nBJ_n of the form

$$\tilde{L}(J_nBJ_n) - (J_nBJ_n)\tilde{L}^{\mathrm{T}} = \tilde{\boldsymbol{q}}\tilde{\boldsymbol{p}}^{\mathrm{T}} - \tilde{\boldsymbol{p}}\tilde{\boldsymbol{q}}^{\mathrm{T}},$$

where $\tilde{L}$ is a lower triangular matrix with unit diagonal entries. Since Bernstein polynomials are symmetric, i.e. $\beta_i^{(n)}(z) = \beta_{n-i}^{(n)}(1-z)$, we find that, up to the sign, J_nBJ_n is the Bernstein–Bezoutian matrix associated with $p(1-z)$ and $q(1-z)$. Therefore, Theorem 2 allows us to reduce the computation of the GCD of $p(z)$ and $q(z)$ to determining a block triangular factorization of J_nBJ_n.

To do this we can consider the generalized Schur algorithm for generalized displacement structures described in [20,21]. The derivation of this algorithm relies upon the fundamental property that the Schur complement of a nonsingular leading principal submatrix of J_nBJ_n inherits the same displacement structure of J_nBJ_n. This enables the elimination procedure to be defined by means of a set of recursions only involving the displacement generators. Although the algorithm presented in [20,21] only works in the strongly nonsingular case, where all the leading principal submatrices of J_nBJ_n are nonsingular, its extension to cover input matrices with singular submatrices is straightforward. In fact, by a continuity argument a block elimination step can be reduced to performing a sequence of steps. Observe that the size of the jumps occurring in the block elimination procedure can be determined by a direct inspection of the computed Schur complements as shown in Example 4.

Summarizing , we apply the generalized Schur algorithm for the block triangular factorization of J_nBJ_n to obtain a fast $\mathrm{O}(n^2)$ algorithm for the evaluation of the GCD of two polynomials $p(z)$ and $q(z)$ given in Bernstein form.

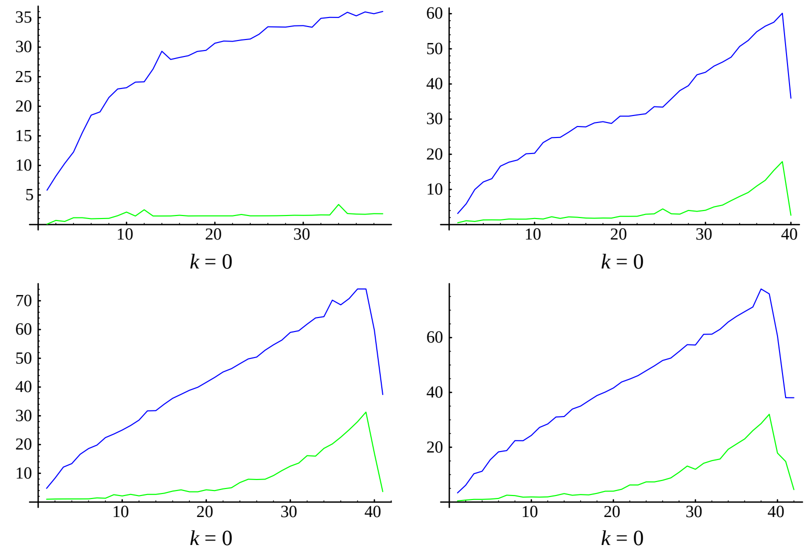

Fig. 1.

4. Numerical experiments

The numerical performance of the LU factorization algorithm applied to a matrix J_nAJ_n is strongly influenced by the condition numbers of the $k \times k$ trailing principal submatrices A_k of A. Therefore, in order to compare the performances of computing the GCD of two polynomials in the monomial basis and in the Bernstein basis, we have compared the values of the condition numbers of the trailing principal submatrices $\tilde{B}_k$ and B_k of $\tilde{B}$ and B, respectively. In fact, $\tilde{B}$ and B are the representations of the Bezoutian $b(z,w) = (p(z)q(w) - p(z)q(w))/(z-w)$ in the monomial basis and in the Bernstein basis, respectively.

For $k = 0, 1, 2, 3$ we have generated two pseudo-random polynomials $p(z)$ and $q(z)$ of degree $n = m + k$, with $m = 40$, having a common factor $s(z)$ of degree k, in the following way: let P_i and Q_i be random integers uniformly distributed in the range $[-100, 100]$, set $p(z) = s(z)\sum_{i=0}^{m} P_i\beta_i^{(m)}(z)$, $q(z) = s(z)\sum_{i=0}^{m} Q_i\beta_i^{(m)}(z)$. The common factor $s(x)$ has been chosen in the set $\{1, z+2, (z+2)(z-3), (z+2)(z-3)(z+1/3)\}$. From $p(z)$ and $q(z)$ we have constructed the matrices $\tilde{B}$ and B.

In Fig. 1, we report the plots of the logarithm to the base 10 of the spectral condition numbers of the matrices $\tilde{B}_i$ (dark grey), and B_i (light grey) for $i = 1, \ldots, m+k$, for the values $k = 0, 1, 2, 3$. As we can see, the growth of the spectral condition number with respect to i is much larger for the Bezoutian represented in the monomial basis than for the Bezoutian represented in the Bernstein basis. In particular, if the polynomials $p(z)$ and $q(z)$ are relatively prime ($k = 0$) then the condition numbers of B_i are uniformly bounded. This shows that any numerical method for the computation of $\mathrm{GCD}(p(z), q(z))$ based on the LU factorization of the Bezout matrix is less prone to

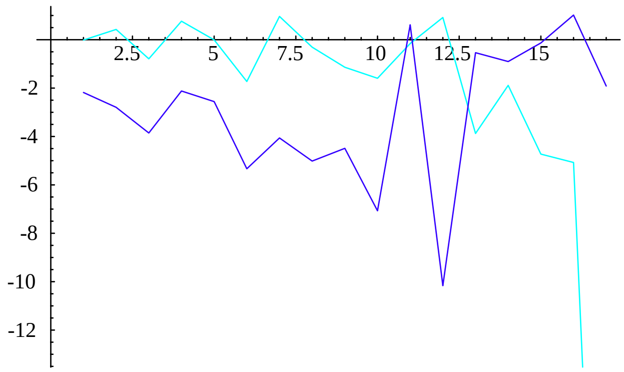

Fig. 2.

numerical instability if the computation is performed in the Bernstein basis rather than in the monomial basis.

We also considered the use of SVD to obtain a satisfactory "approximate-gcd" for polynomials in Bernstein form (see [22] and the bibliography therein for a review of some major known methods for "approximate-gcds" of polynomials in power form). In [9] it was proved that a reasonable termination criterion for the Euclid's algorithm in floating-point arithmetic is to test the ratio between the smallest singular values of two consecutive submatrices of the subresultant of the input polynomials against a prescribed tolerance. In Fig. 2 we compare the robustness of this indicator for Bernstein–Bezoutian matrices (light grey) and classical Bezoutian matrices (dark grey) obtained after having performed the explicit conversion between the Bernstein and the power basis. Specifically, we plot the logarithm to the base 10 of the ratio between the smallest singular values of two consecutive leading principal submatrices. The input polynomials are pseudo random polynomials of degree 20 in Bernstein form with a common factor of degree 1. Computations are carried out using the standard numerical precision of about 16 digits. We see that for classical Bezout matrices the ratio profile experiences dramatic and unpredictable changes at each successive stage so that its comparison with a specified tolerance is an unreliable indicator of when to stop the Euclidean algorithm. On the contrary, the test performs quite well for Bernstein–Bezoutian matrices.

5. Future work

The results presented in this paper provide theoretical bases for the design of fast and accurate algorithms for the computation of the GCD of two polynomials in Bernstein form. Our future research will mainly focus on studying the numerical behavior of these algorithms.

A look-ahead strategy can be incorporated into the generalized Schur algorithm in order to improve its robustness and accuracy. An implementation of the Schur algorithm in a look-ahead fashion, using variable-precision floating point arithmetic, would provide a competitive method for solving the GCD problem for polynomials in the Bernstein basis.

Schur algorithm is based on the invariance of the Bezoutian structure under Schur complementation. This property, rephrased into a polynomial setting, leads to polynomial schemes for the computation of the GCD of two given polynomials. We refer to [2,3] for a description of these schemes for polynomials expressed with respect to the standard power basis. In particular in [3] it was noticed that the polynomial equivalence can be exploited in order to decrease the Boolean complexity of the factorization procedures.

The results of this paper allow us to extend the approach in [2,3] to the case where the input polynomials are represented in the Bernstein basis. In this way, we obtain polynomial remainder algorithms for computing the GCD of two polynomials in Bernstein form which retain the Bernstein basis throughout the computation. Extensive numerical experiments comparing the Boolean cost of these polynomial schemes and of the generalized Schur algorithm would yield some important insights in order to arrive at a conclusive choice.

Besides GCD computation, the properties of Bezoutian matrices allow the design of efficient root localization procedures and stability tests for scalar and matrix polynomials. This transfers also to Bernstein–Bezoutian matrices and the study of the resulting algorithms would be useful for a variety of applications in computer graphics.

Acknowledgements

We wish to thank the anonymous referees whose valuable suggestions greatly improved the presentation of the paper. This work was partially supported by MIUR, Grant number 2002014121.

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